21-356 Homework 6

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1

Let $A \subset \mathbb{R}^N$ be an open set and let $f : A \to \mathbb{R}^N$ be a function of class C^1 with $\det Jf(x) \neq 0$ for all $x \in A$. Given a point $y_0 \notin f[A]$, let

$$g(x) = |y_0 - f(x)|^2 \quad x \in A$$

Show that $\nabla g(x) \neq 0$ for all $x \in A$.

First, note that f maps A into $\mathbb{R}^N$, and since $y_0 \notin f[A]$, $g(x)$ is always (the square of) a positive number. We need to show that $f(x)$ never runs "parallel to y_0" — that is to say, never runs tangent to any hypersphere centered at y_0.

Since $\det Jf(x) \neq 0$, by the inverse function theorem we know that f is invertible near x, for any $x \in A$, and therefore that f is entirely invertible.

2

Given the equation

$$a \log(1 + xy) + a^2 xy - 2 \sin x + y - 1 = 0,$$

where $a \in \mathbb{R}$,

(a.) *prove that in a neighborhood of the point $P = (0, 1)$, the equation implicitly determines y as a function of x.*

The Jacobian of this function at $(a, x, y) \in \mathbb{R}^3$ is

$$Jf(a, x, y) = \big(\log(1 + xy) + 2axy \quad ay(1 + xy)^{-1} + a^2 y - 2 \cos x \quad ax(1 + xy)^{-1} + a^2 x + 1\big)$$

which at $P = (a, 0, 1)$ is $(0, a^2 + a - 2, 1)$. In other words, as y increases by 1, x increases by $a^2 + a - 2$. So we have $x \approx (a^2 + a - 2)(y - 1)$, or $y \approx x/(a^2 + a - 2) + 1$. (This function indeed passes through $(a, 0, 1)$, which is a good sign.) *Q.E.D.*

(b.) *Determine, if they exist, the values of a for which the equation determines a function $y = y(x)$ with a maximum at the point $x = 0$.*

Since the implicitly determined function is *linear*, it can't possibly have a maximum anywhere! This implies that our answer to part (a) must have been wrong.

(c.) *Sketch the graph of the function $y = y(x)$ near the point $x = 0$.*

3

Given the function $f(x,y,z,w) = (y^2 + w^2 - 2xz, \quad y^3 + w^3 + x^3 - z^3)$,

(a.) *prove that in the neighborhood of* $(1,-1,1,1)$, *the hypotheses of the implicit function theorem are satisfied and that there exists a function* $g(y,w) = (g_1(y,w), g_2(y,w))$ *such that*

$$f(g_1(y,w), y, g_2(y,w), w) = (0,0)$$

for all (y,w) *near* $(-1,1)$.

We need to show that $\det Jf(y,w,x,z) = \det \frac{f_1, f_2}{\partial y,w,x,z} \neq 0$. That's easy:

$$\nabla f_1 = (2y, 2w, -2z, -2x)$$
$$\nabla f_2 = (3y^2, 3w^2, 3x^2, -3z^2)$$

and $x, y, z, w \neq 0$, so we win.

By the implciit function theorem, then, there exists a $g = (g_1, g_2)$ of the kind we're looking for, and it expresses x, z in terms of y, w.

(b.) *find the first-order partial derivatives of* g *at the point* $(-1,1)$.

We can find these derivatives by implicit diferenation. Example:

$$(y^2 + w^2 - 2xz, \quad y^3 + w^3 + x^3 - z^3) = 0$$
$$y^2 + w^2 = 2xz$$
$$y^3 + w^3 = x^3 - z^3$$

4

Given the unction

$$f(x,y,z) = (z\cos xy, \ z\sin xy, \ x+z),$$

(a.) *prove that* f *has a local inverse in the heighboord of* $(1,0,1)$.

That is, prove that f is invertible. That's the same as showing that f is C^1 (which is obvious), and then that its Jacobian determinant is non-zero everywhere.

$$Jf(x,y,z) = \begin{pmatrix} -yz\sin xy & yz\cos xy & 1 \\ -xz\sin xy & xz\cos xy & 0 \\ \cos xy & \sin xy & 1 \end{pmatrix}$$

The determinant is

$$(-yz\sin xy)(xz\cos xy) - (yz\cos xy)(-xz\sin xy) - xz = -xz$$

which is non-zero at $x = 1, z = 1$. *Q.E.D.*

(b.) *find the Jacobian matrix of* f^{-1} *at the point* $(1,0,2)$.

Use a linear approximation of f^{-1}: we know the Jacobian is

$$Jf(x,y,z) = \begin{pmatrix} -yz\sin xy & yz\cos xy & 1 \\ -xz\sin xy & xz\cos xy & 0 \\ \cos xy & \sin xy & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

So we have $f(x,y,z) \approx (z,\ y,\ x+z)$; so we can write the inverse as $f^{-1}(a,b,c) \approx (c-a,\ b,\ a)$. So the Jacobian of that is simply

$$Jf^{-1}(a,b,c) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Q.E.D.