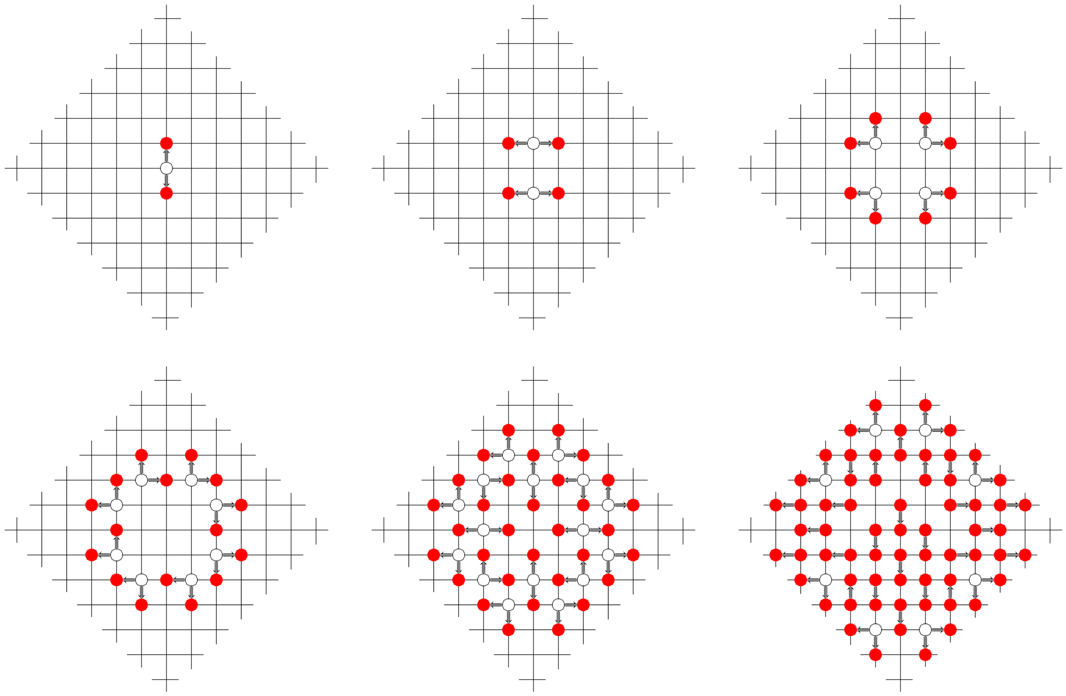
# 1 The coin game.

The game must come to an end in at most six turns. To prove this, we show a game of six turns, and then prove that no game can take longer.



Now, no game can take more than six turns, since the "speed of light" in this game is just one space per turn — that is, immediately after the $n$th move, no coin can possibly be further than $n$ spaces away from the origin. Thus, in $n$ turns the most coins that can have been placed is just $n^2 + (n+1)^2$, or $2n^2 + 2n + 1$. But we must place $2^n$ coins on the $n$th turn! So, we must have failed to place all the coins (at *most*) by the time $2^n > 2n^2 + 2n + 1$.

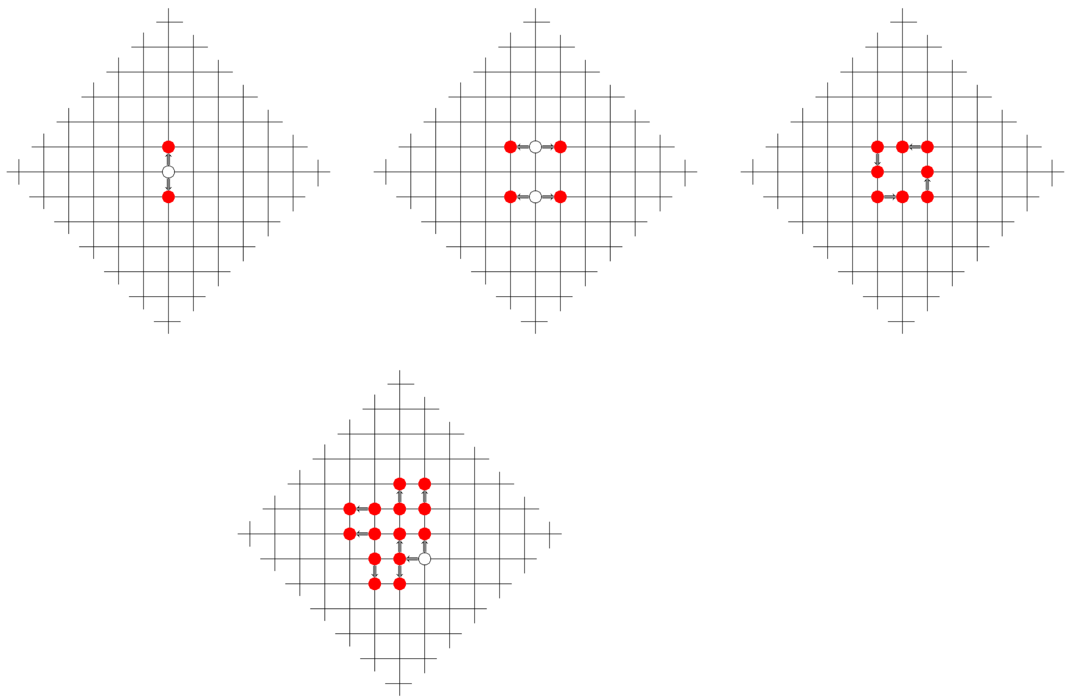
| $n$ | $2^n$ | $2n^2 + 2n + 1$ |
|---|---|---|
| 1 | 2 | 5 |
| 2 | 4 | 13 |
| 3 | 8 | 25 |
| 4 | 16 | 41 |
| 5 | 32 | 61 |
| 6 | 64 | 85 |
| 7 | 128 | 113 |

Thus, six moves is the most we can hope for. And, as we have shown, there is a game of six moves. *Q.E.D.*

# 2 Misère coin game.

Suppose that instead of asking for "a lower bound for the largest number of complete turns," we ask for a lower bound on the *smallest* number of complete turns in a finished game. What is the answer then?

The answer is four moves. The following game has no legal fifth move:



We can see that fact clearly by observing that the figure resulting from the fourth move has a "perimeter" of only 15 spaces; thus, there's nowhere to place another 16 coins. This is also a *shortest* complete game, since no figure containing only eight coins has a perimeter less than 8, nor any four-coin figure a perimeter less than 4. (That can be proved by brute force.)